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COMPUTATIONAL-THEORETICAL INVESTIGATION OF BUOYANT JET FLOWS

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The results of mathematical simulation of fully developed plane buoyant jet flows are presented. The solution is found within the framework of the model of a laminar boundary layer in the Boussinesq approximation using the method of matched asymptotic expansions. Analytical dependences of the basic characteristics of a jet flow on the Prandtl number and density parameter have been constructed.

Keywords: *vertical plane jet flows, free-convective heat transfer, laminar regime, mathematical simulation.*

Introduction. The problem of studying the jet transfer of momentum and heat in the field of buoyancy forces is of interest from the viewpoint of studying physical processes occurring in buoyant jet flows as well as in the context of their use in thermal-power engineering machines and apparatuses. However, for designing and refining power equipment it is important to develop calculation methods that could be of general character and allow for the joint effect of various factors on jet development. Furthermore, for engineering practice it is important that the obtained computational relations reflect the physics of the phenomenon and be simple and convenient for computations.

To date, abundant information [1–6] has already been gathered on the hydrodynamics and heat transfer in buoyant jets in the case of the linear dependence of density on temperature. Buoyant motions of liquid above heat sources in the case of nonlinear dependence of density on temperature remain among the most inadequately studied types of jet motions. The available information on this topic is very insufficient [7–9] despite the obvious practical importance of these data.

In what follows, we present the results of a complex analytical and numerical investigation of fully developed buoyant flows over a horizontal linear heat source within the framework of the model of a laminar boundary layer in the Boussinesq approximation.

Statement of the Problem. We consider the regime of a stationary laminar vertical flow of liquid over an infinitely long, horizontal, heated fine wire in a pool of liquid. We introduce the Cartesian coordinate system x, y with the corresponding velocity components u, v. The axis is directed vertically upward in the symmetry plane of the jet. The physical properties of the liquid are considered constant except for the density $\rho = \rho_\infty[1 - \beta_q(\Delta T)^q]$. The initial equations are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta_q(\Delta T)^q ,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \qquad (1)$$

$$u\frac{\partial \Delta T}{\partial x} + v\frac{\partial \Delta T}{\partial y} = \frac{\nu}{\mathrm{Pr}}\frac{\partial^2 \Delta T}{\partial y^2} .$$

The boundary condition for the given problem have the form

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$$y=0: \quad v=\frac{\partial u}{\partial y}=\frac{\partial \Delta T}{\partial y}=0\,, \qquad y\to\infty: \quad u\to 0\,, \quad \Delta T\to 0\,. \tag{2}$$

The statement is completed by writing the initial conditions for the velocity and temperature fields of the jet. However, since we are interested in the character of the change in the basic characteristics only for the self-similar region of jet flow development, we replace the initial conditions by the integral relation

$$Q_0=\rho C_p\int_{-\infty}^{+\infty} u\Delta T dy\,, \tag{3}$$

which follows from the third equation of system (1) and is the condition for the preservation of the value of Q_0 in any horizontal plane x = const over a heat source. We introduce into consideration the stream function $\Psi(x, y)$ from the relations $u = \partial\Psi/\partial y$ and $v = -\partial\Psi/\partial x$. Then the second equation of system (1) is satisfied identically. We will seek the solution of the system of equations (1)–(3) in the form

$$\Psi=\left[g\beta_q\nu^2\left(\frac{Q_0}{\rho C_p}\right)^q\right]^{\frac{1}{4+q}} f(\eta)\, x^{\frac{3}{4+q}}\,,$$

$$\eta=\left[\frac{g\beta_q}{\nu^2}\left(\frac{Q_0}{\rho C_p\nu}\right)^q\right]^{\frac{1}{4+q}} yx^{-\frac{1+q}{4+q}}\,,$$

$$u=\left[g\beta_q\left(\frac{Q_0}{\rho C_p\nu^{1/2}}\right)^q\right]^{\frac{2}{4+q}} f'(\eta)\, x^{\frac{2-q}{4+q}}\,, \tag{4}$$

$$\Delta T=\left[\frac{Q_0^4\nu^2}{g\beta_q(\rho C_p\nu)^4}\right]^{\frac{1}{4+q}} h(\eta)\, x^{-\frac{3}{4+q}}\,.$$

Then in the self-similar approximation we obtain ordinary differential equations to determine the unknown functions $f(\eta)$ and $h(\eta)$ (the derivative with respect to η is primed):

$$f'''+\frac{3}{4+q}ff''-\frac{2-q}{4+q}f'^2+h^q=0\,,$$

$$\frac{1}{\mathrm{Pr}}h''+\frac{3}{4+q}(fh)'=0\,, \tag{5}$$

$$f(0)=f''(0)=h'(0)=0\,,$$

$$f'(\infty)=h(\infty)=0\,, \quad 2\int_0^{+\infty} f'hd\eta=1\,.$$

It follows from Eqs. (4) and (5) that the behavior of a buoyant jet is determined by two quantities: the Pr number and the density parameter q. In this case, the influence of the dependence of density on temperature is manifested via a change in the pattern of the physical process studied, in particular, in the law of increase (decrease) of u, v, ΔT with the height x. We will carry out an analysis for an arbitrary number q, whereas the cases of $q = 1$ [10] and $q = 2$ [9] are considered as particular ones. The solution of the system of equations (5) will be made by the method of matched asymptotic expansions, making it possible to promptly determine the joint influence of various factors on the hydrodynamics and heat transfer of a jet flow with an accuracy sufficient for practice.

Solution of the Problem. Let Pr >> 1. In this case, the thermal boundary layer is thinner than the dynamic one and the influence of buoyant forces will be exhibited only in the region of the thermal layer. This important fact allows us to introduce into consideration a two-layer mathematical model, as well as new functions and variables for the inner and outer layers:

$$f(\eta) = \mathrm{Pr}^{\frac{5}{2(q+4)}} F(\zeta), \quad \zeta = \eta \mathrm{Pr}^{\frac{3+2q}{2(q+4)}}, \quad h(\eta) = \mathrm{Pr}^{-\frac{5}{2(q+4)}} H(\zeta); \tag{6}$$

$$f(\eta) = \mathrm{Pr}^{\frac{q-1}{2(q+4)}} G(z), \quad z = \eta \mathrm{Pr}^{\frac{q-1}{2(q+4)}}, \quad h(\eta) = 0 .$$

The substitution of (6) into (5) yields

$$F''' + \frac{1}{\mathrm{Pr}}\left(\frac{3}{4+q} FF'' - \frac{2-q}{4+q} F'^2\right) + \frac{1}{\mathrm{Pr}^{1/2}} H^q = 0 ,$$

$$H' + \frac{3}{4+q} FH = 0 , \quad \int_0^{+\infty} F' H d\zeta = \frac{1}{2} , \tag{7}$$

$$G''' + \frac{3}{4+q} GG'' - \frac{2-q}{4+q} G'^2 = 0 ,$$

where the prime means differentiation with respect to the corresponding variables ζ and z. The quantity $f'(0)$ characterizes a change in the axial values of velocity u, $h(0)$ — in the temperature ΔT, and $f(\infty)$ — in the flow rate m in the jet. It follows from relations (6) that the temperature of a buoyant jet increases with the Prandtl number according to the power law with the exponent $5/(2q+8)$. There is a similar tendency also for the arbitrary velocity component $(q-1)/(q+4)$. The mass flow rate of liquid in a jet also increases according to the power law with the exponent $(q-1)/(2q+8)$.

The functions F, H, and G will be determined in the form of asymptotic series. Since internal and external expansions mutually influence each other, they must contain identical exponents ε. The quantity $\varepsilon = \mathrm{Pr}^{-1/2}$ is used as a small parameter: it is seen from (7) that no series in other degrees of the Pr number can appear:

$$F(\zeta) = F_0(\zeta) + \varepsilon F_1(\zeta) + \varepsilon^2 F_2(\zeta) + \ldots.. ,$$

$$H(\zeta) = H_0(\zeta) + \varepsilon H_1(\zeta) + \varepsilon^2 H_2(\zeta) + \ldots.. , \tag{8}$$

$$G(z) = G_0(z) + \varepsilon G_1(z) + \varepsilon^2 G_2(z) + \ldots.. .$$

After the substitution of (8) into (7) and splitting of the resulting system by the parameter ε, we arrive at an infinite system of equations for the unknown functions F_j, H_j, and G_j that determine the solutions of the internal and external problems. It follows from (7) that these solutions are connected only through the boundary conditions. Therefore additionally it is necessary to formulate the rules of the "sewing" of asymptotic series. According to the agreement conditions, the following equalities should be fulfilled:

$$\lim_{\zeta\to\infty}(F_0+\varepsilon F_1+\varepsilon^2 F_2+.....)=\lim_{z\to 0}(\varepsilon G_0+\varepsilon^2 G_1+\varepsilon^3 G_2+.....)\,,$$

$$\lim_{\zeta\to\infty}(H_0+\varepsilon H_1+\varepsilon^2 H_2+.....)=0\,. \tag{9}$$

As a result, restricting ourselves to three approximations, we obtain:

the internal problem

$$F_0'''=0\,,\quad H_0'+\frac{3}{4+q}F_0H_0=0\,,\quad \int_0^\infty F_0'H_0 d\zeta=1/2\,,$$

$$F_0(0)=F_0''(0)=F_0''(\infty)=H_0(\infty)=0\,,$$

$$F_1'''+H_0^q=0\,,\quad H_1'+\frac{3}{4+q}F_0H_1=-\frac{3}{4+q}F_1H_0\,,$$

$$\int_0^\infty (F_0'H_1+F_1'H_0)\,d\zeta=0\,,$$

$$F_1(0)=F_1''(0)=H_1(\infty)=0\,,\quad F_1''(\infty)=G_0''(0)\,,$$

$$F_2'''+qH_0^{q-1}H_1=-\frac{3}{4+q}F_0F_0''+\frac{2-q}{4+q}F_0'^2, \tag{10}$$

$$H_2'+\frac{3}{4+q}F_0H_2=-\frac{3}{4+q}F_1H_1-\frac{3}{4+q}F_2H_0\,,$$

$$\int_0^\infty (F_0'H_2+F_1'H_1+F_2'H_0)\,d\zeta=0\,,$$

$$F_2(0)=F_2''(0)=H_2(\infty)=0\,,\quad F_2''(\infty)=G_1''(0)\,;$$

. . . .

the external problem

$$G_0'''+\frac{3}{4+q}G_0G_0''-\frac{2-q}{4+q}G_0'^2=0\,,$$

$$G(0)=G_0'(\infty)=0\,,\quad G_0'(0)=F_0'(\infty)\,,$$

$$G_1'''+\frac{3}{4+q}G_0G_1''-\frac{4-2q}{4+q}G_0'G_1'+\frac{3}{4+q}G_0''G_1=0\,,$$

$$G_1(0)=\lim_{\zeta\to\infty}(F_0-\zeta G_0'(0))\,, \tag{11}$$

$$G_1'(0) = \lim_{\zeta\to\infty} (F_1' - \zeta G_0''(0)) , \quad G_1'(\infty) = 0 ,$$

$$G_2''' + \frac{3}{4+q} G_0 G_2'' - \frac{4-2q}{4+q} G_0' G_2' + \frac{3}{4+q} G_0'' G_2 = -\frac{3}{4+q} G_1 G_1'' + \frac{2-q}{4+q} G_1'^2 ,$$

$$G_2(0) = \lim_{\zeta\to\infty} \left(F_1 - \zeta G_1'(0) - \frac{\zeta^2}{2} G_0''(0) \right) ,$$

$$G_2''(0) = \lim_{\zeta\to\infty} (F_2' - \zeta G_1''(0)) , \quad G_2'(\infty) = 0 ;$$

. . . .

A distinctive feature of Eqs. (10)–(11) as compared to (5) is the presence of complex connections that take account of the interaction of the inner and outer boundary layers. Therefore the solution of Eqs. (10)–(11) is reduced to integration by steps of the internal and external problems.

The zero approximation (8) is written in quadratures:

$$F_0 = c_2 \zeta , \quad H_0 = c_4 \exp\left(-\frac{3c_2}{2(4+q)} \zeta^2 \right) , \quad c_4 = \left(\frac{3}{2(4+q) c_2 \pi} \right)^{1/2} . \tag{12}$$

The expression for the function F_1 has the form

$$F_1 = -\frac{c_4^q}{2} \zeta^2 \int_0^\zeta \exp\left(-\frac{3c_2 q}{2(4+q)} \zeta^2 \right) d\zeta - c_4^q \frac{4+q}{6c_2 q} \left(\zeta \exp\left(-\frac{3c_2 q}{2(4+q)} \zeta^2 \right) \right.$$

$$\left. + \int_0^\zeta \exp\left(-\frac{3c_2 q}{2(4+q)} \zeta^2 \right) d\zeta \right) + c_6 \zeta . \tag{13}$$

Knowing F_1 and H_0, we find H_1:

$$H_1 = -\frac{3}{4+q} c_4 \exp\left(-\frac{3c_2}{2(4+q)} \zeta^2 \right) \left(-\frac{c_4^q}{6} \zeta^3 \int_0^\zeta \exp\left(-\frac{3c_2 q}{2(4+q)} \zeta^2 \right) d\zeta \right.$$

$$- c_4^q \frac{4+q}{18c_2 q} \zeta^2 \exp\left(-\frac{3c_2 q}{2(4+q)} \zeta^2 \right) - \frac{c_4^q (4+q)^2}{27 c_2^2 q^2} \exp\left(-\frac{3c_2 q}{2(4+q)} \zeta^2 \right)$$

$$\left. - c_4^q \frac{4+q}{6c_2 q} \zeta \int_0^\zeta \exp\left(-\frac{3c_2 q}{2(4+q)} \zeta^2 \right) d\zeta + c_6 \frac{\zeta^2}{2} + \frac{c_4^q (4+q)^2}{3c_2^2 \sqrt{1+q}} \left(\frac{c_2 c_6 \sqrt{1+q}}{2 c_4^q (4+q)} + \frac{(1+q)(1-2q)}{9q^2} \right) \right) . \tag{14}$$

The coefficients c_2 and c_6 are undetermined as yet. They are calculated further in the process of solution. Now, in conformity with the scheme of the problem analysis it is necessary to consider a jet flow of liquid in the outer region. Taking into account Eqs. (12) and (13), we will have

$$G_0''' + \frac{3}{4+q} G_0 G_0'' - \frac{2-q}{4+q} G_0'^2 = 0 ,$$

$$G(0)=0\,,\quad G_0'(0)=c_2\,,\quad G_0'(\infty)=0\,;$$

$$G_1'''+\frac{3}{4+q}\,G_0G_1''-\frac{2\,(2-q)}{4+q}\,G_0'G_1'+\frac{3}{4+q}\,G_0''G_1=0\,, \tag{15}$$

$$G_1(0)=0\,,\quad G_1'(0)=c_6\,,\quad G_1'(\infty)=0\,.$$

The system of equations (15) allows no analytical solution, and therefore it becomes necessary to use numerical approaches. The functions G_0 and G_1 were found by the Runge–Kutta method by reducing (15) to the corresponding Cauchy problems. A numerical analysis has shown that the behavior of the sought functions is very sensitive to the lacking boundary conditions at $z = 0$. The latter leads to inevitable errors in calculations and is reflected substantially on the accuracy of numerical solutions. The emerging difficulties are obviated in the present work by using the method of preliminary search for unknown initial parameters. The adopted method can be interpreted as an iteration process for constructing a solution in the form of a series: the functions G_0 (G_1) are decomposed into n terms, with the solution corresponding to the case $G_0''' = 0$ being taken as the zero approximation. Next, three (four) terms of the series are constructed, precisely with the aid of which the formulas for the estimated values of c_2 and c_6 are found. The simple but cumbersome calculations carried out by the described algorithm made it possible to obtain an exact coupling equation and the value of c_2 at some of the values of the parameter q:

$$c_6=\frac{2c_4^q\,(2\sqrt{2}+(2q-1)\,\sqrt{1+q})}{9c_2q}\,,$$

$$q=1:\ \ c_2=\left(\frac{782}{929}\right)^{2/5},\quad q=\frac{4}{3}:\ \ c_2=\frac{1}{10}\left(\frac{689\sqrt{6}}{27}\right)^{1/2}, \tag{16}$$

$$q=\frac{3}{2}:\ \ c_2=\left(\frac{625}{1099}\right)^{6/11},\quad q=2:\ \ c_2=\left(\frac{311}{656}\right)^{2/3}.$$

On the basis of (16) it is possible to approximate the dependence of c_2 on q by a function of the form

$$c_2=\left(\frac{q^2}{\frac{431}{595}-\frac{788}{273}\,q+\frac{2846}{855}\,q^2+\frac{107}{5000}\,q^3}\right)^{\frac{2q}{4+q}}. \tag{17}$$

The results obtained agree well with the data of numerical integration. The values of $G_0'(0) = F_0'(0) = c_2$ and $G_1'(0) = c_6$ found with the use of a numerical scheme (see Table 1) and relations (16) virtually coincide: the maximum absolute error is of the order of $1{\cdot}10^{-7}$.

Next we determine $f'(0)$ and $h(0)$:

$$f'(0)=\mathrm{Pr}^{\frac{q-1}{q+4}}\,(F_0'(0)+\mathrm{Pr}^{-1/2}F_1'(0)+....)=\mathrm{Pr}^{\frac{q-1}{q+4}}\left(c_2+\mathrm{Pr}^{-1/2}\left(c_6-\frac{c_4^q\,(4+q)}{3c_2q}\right)+...\right),$$

$$h(0)=\mathrm{Pr}^{\frac{5}{2(q+4)}}\,(H_0(0)+\mathrm{Pr}^{-1/2}H_1(0)+...) \tag{18}$$

TABLE 1. Results of Numerical Calculation of the Values of Functions $F_0'(0)$, $F_1'(0)$, $H_0(0)$, $H_1(0)$, $G_0(\infty)$, and $G_1(\infty)$

q	$F_0'(0)$	$-F_1'(0)$	$H_0(0)$	$H_1(0)$	$G_0(\infty)$	$G_1(\infty)$
1	0.9334186	0.2480414	0.3198508	0.1021349	1.3559608	0.2346581
4/3	0.7906159	0.1295526	0.3365027	0.0929766	1.3095232	0.2196044
1.5	0.7350205	0.0917443	0.3436684	0.0880120	1.2927936	0.2139360
2	0.6080012	0.0233309	0.3617786	0.0751745	1.2601647	0.1989087

$$= \Pr^{\frac{5}{2(q+4)}} \left(c_4 - \Pr^{-1/2} \frac{c_4^{q+1}}{9c_2^2 q^2} [(2\sqrt{2} - 1)\, q + (4 - 8q)\sqrt{1+q} - 4] + ... \right).$$

The basic characteristic feature of (18) consists of different asymptotic behaviors of the first and second terms. This allows one to assume that expression (18) obtained on condition that Pr >> 1 can also be used at moderate values of Pr. We will analyze the case with $q = 1$:

$$h(0) = 0.3198508\Pr^{1/2} + 0.1021349 + ... \,. \tag{19}$$

A comparison of the value of $h(0)$ with the data of numerical integration of the system of equations (5) [11] shows that if the 2%-difference is taken as a criterion, the agreement begins at Pr = 0.3.

To elucidate the limits of applicability of the solution constructed, a comparison with the results of [7] was also carried out. Calculations were performed at the following values of the quantities entering into Eq. (18): four values of the parameter q = 1.58295, 1.72715, 1.85966, and 1.89482 and the Prandtl number changing from 8.6 to 13.6 with an interval of 1.0. The maximum error is less than 2% for velocity and 1% for temperature. Next we also give the dependence of the mass flow rate in the jet m on q and Pr. By definition, this characteristic is given by the formula

$$m = 2\rho \left[g\beta_q \nu^2 \left(\frac{Q_0}{\rho C_p} \right)^q \right]^{\frac{1}{4+q}} f(\infty)\, x^{\frac{3}{4+q}} , \tag{20}$$

$$f(\infty) = \Pr^{\frac{q-1}{2(q+4)}} (G_0(\infty) + G_1(\infty) \Pr^{-1/2} + ...) .$$

The calculated values of $f(\infty)$ were compared with the results obtained in [8] by the method of numerical integration. The parameter q was taken equal to 2 and the Prandtl number was given the values 6.7, 7, 10, 11.4, and 100. The maximum error of Eq. (20) was 1.7%.

Conclusions. The analytical dependences given in the present work make it possible to rapidly determine the influence of various regime parameters on flow and heat transfer in a vertical plane jet with an accuracy sufficient for practice. This indicates the high efficiency of the method of matched asymptotic expansions as a tool in the solution of practical (research or technological) problems connected with the study of the hydrodynamics and heat transfer in free and mixed convection.

NOTATION

C_p, heat capacity at constant pressure, J/(kg·K); c, constant; g, free-fall acceleration, m/s^2; m, second mass flow rate, kg/(m·s); Pr, Prandtl number; Q_0, flux of excessive heat content, J/(m·s); q, parameter of density; T, temperature, K; u, v, longitudinal and transverse velocity components, m/s; x, y, longitudinal and transverse coordinates, m; β_q, temperature coefficient, 1/K^q; $\Delta T = T - T_\infty$, excessive temperature, K; ν, coefficient of kinematic viscosity, m^2/s; ρ, density, kg/m^3. Subscripts: ∞, surrounding medium.

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